Physics and Lean

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A Vision

A repository like Mathlib but for physics

containing

Fundamental theorems and defs covering physics

Interface with programs, and simulations.

Interface between experimental data and formal theory

Extensive physics based documentation - pedagogy

Computable and efficient definitions

AI?

The repo should

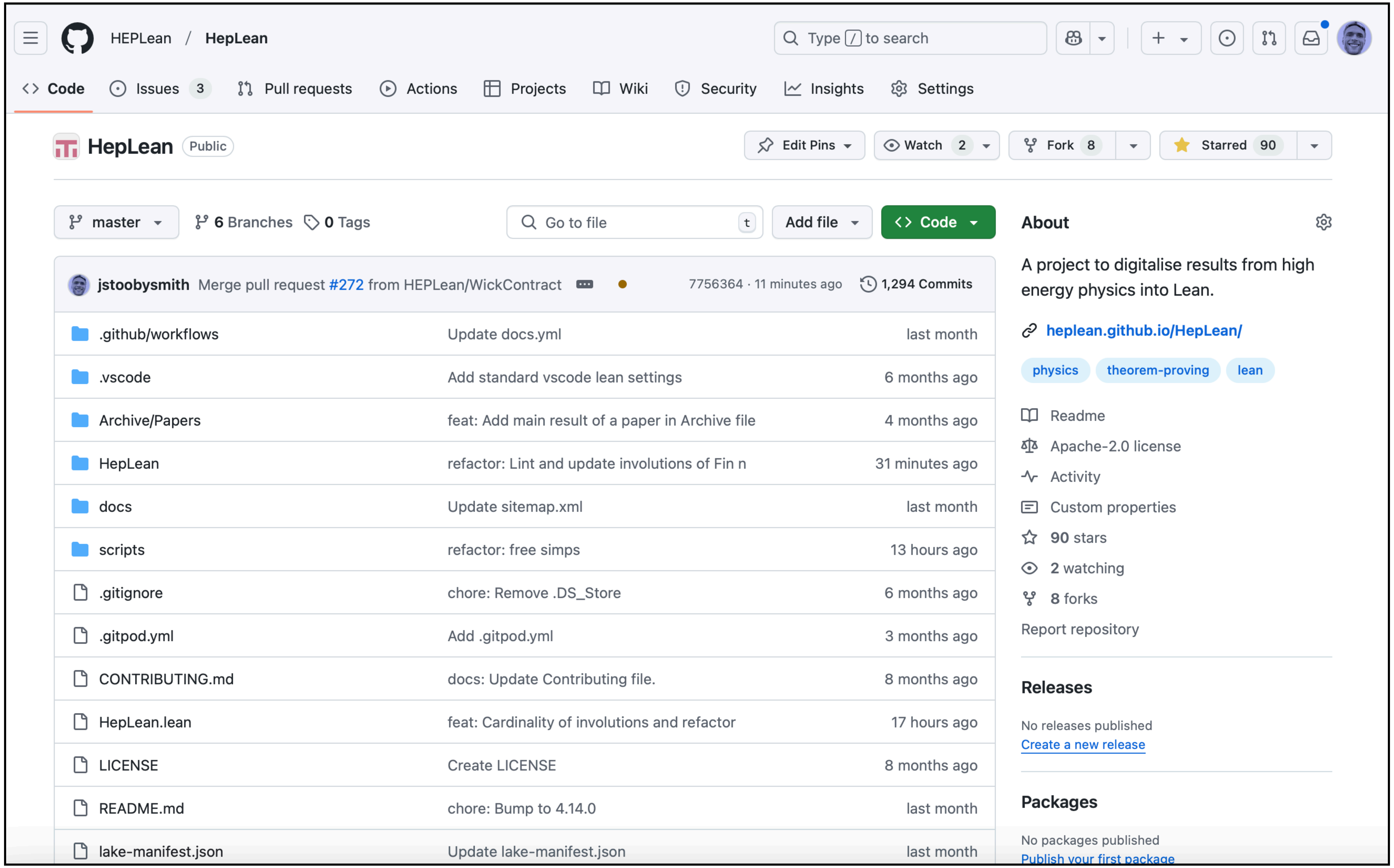
Be organized by physics principals

Be easily searchable akin to loogle, leansearch.net

Look like physics

Be useful to the uninitiated

HepLean



An attempt at this!

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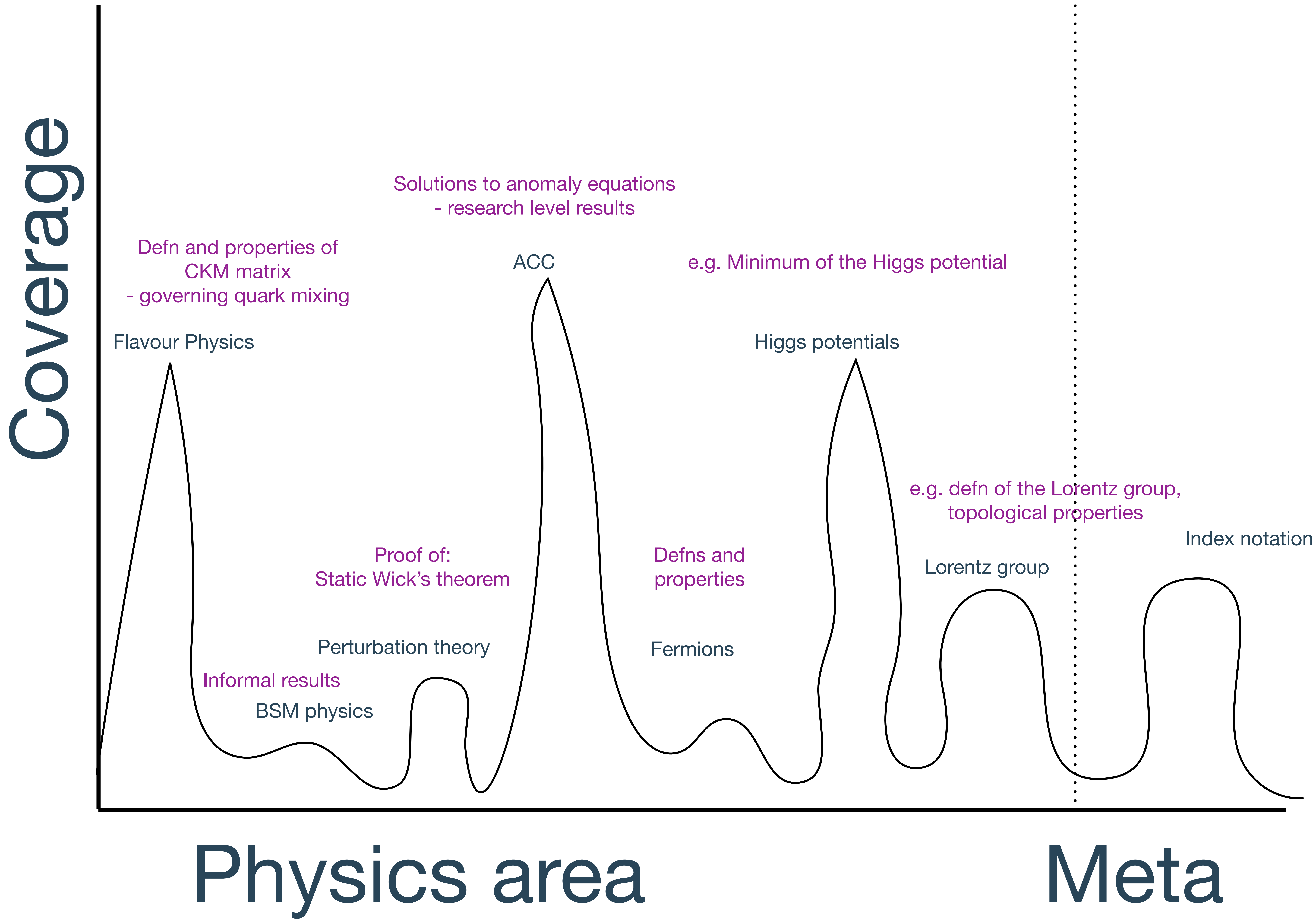
~1000 defns

~2000 Theorems



What is the current progress?

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Is this as easy as for math?

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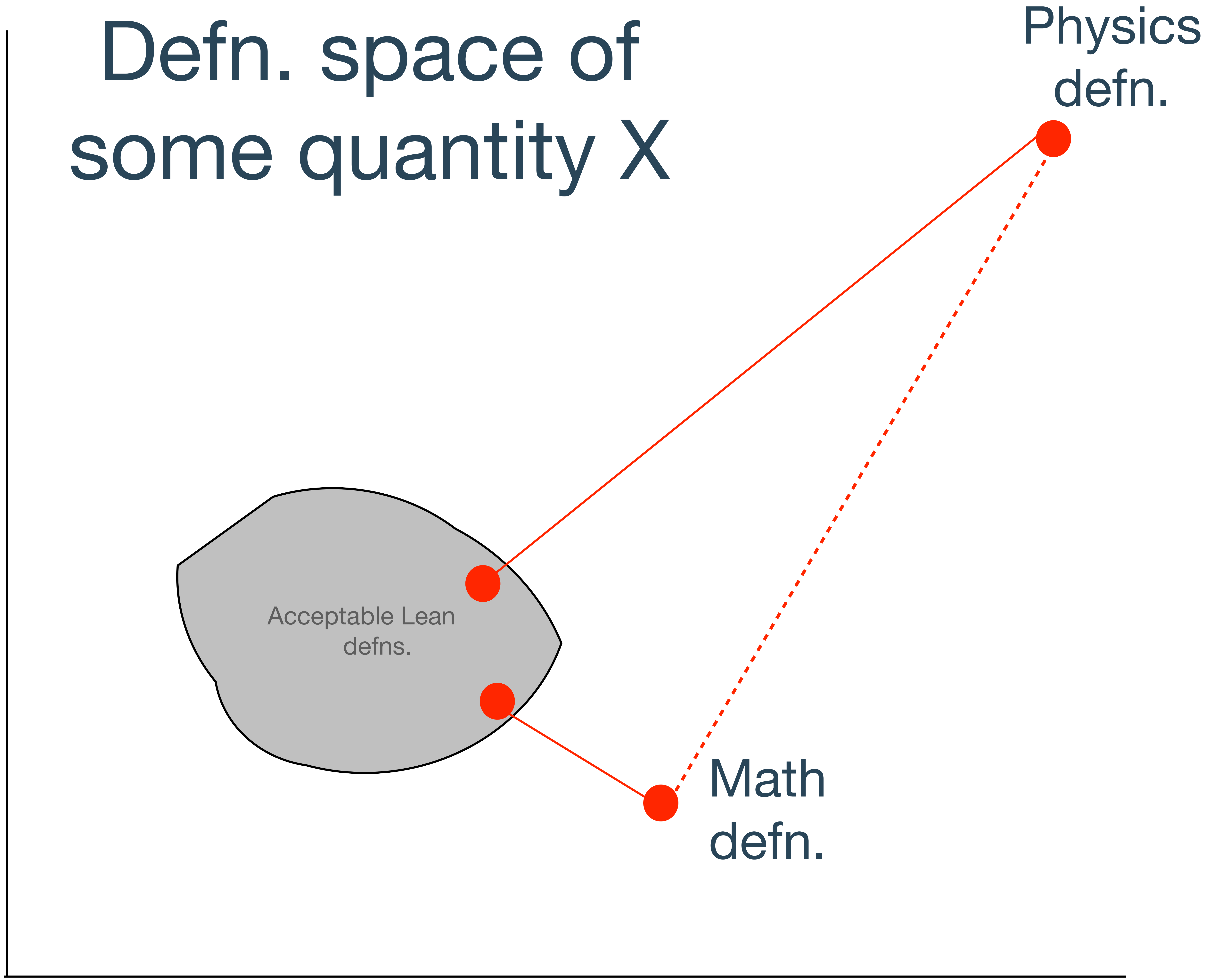
Origin of difficulties

PHYSICS ≠ MATHEMATICS

PHYSICS COMMUNITY ≠
MATHEMATICS COMMUNITY



Definition Distance

- For illustration purposes only - nothing here is well defined.

Paper styles

∼Imprecise results

Specifics not generalties

Long computational and experimental proofs

Challenges

Aut X and $\Omega_x X$ is subtle. Passing to an extrinsic notion of symmetry is convenient in that it allows us to avoid having to do any of this.

To remove some of the mystery of these assertions, consider the simple example of a quantum mechanical system whose dynamics is invariant under rotations of 3-dimensional space. There is an intrinsic $\mathsf{SO}(3)$ symmetry, and one possibility is for the states of the system to form a representation thereof. But the fact that Aut X is non-trivial for the object of quantum-mechanical theories (cf. §4.2) means that one can also have projective representations of $\mathsf{SO}(3)$.

Carrying on our discussion of the physics interpretation, an important remark is that physicists are presumably interested only in objects in X that admit a point (since otherwise there is no QFT in the 'object of QFTs') and only in G-actions on X that admit a homotopy fixed point (since otherwise there will be no QFT that admits a G-symmetry under the given action). Moreover, given a G and an X, it would be desirable to have a means of considering, in one go, all possible actions admitting fixed points, along with their homotopy fixed points. We can achieve this by considering not just the topos $\mathfrak{X}_{/BG}$ in which an object corresponds to a G-action, but also the related category of pointed objects.

Given any topos $\mathfrak{X}$, we define the corresponding category of pointed objects $\mathfrak{X}_*$ as the slice category $\mathfrak{X}_{*/}$.[5] We claim that $(\mathfrak{X}_{/BG})_*$ is a category whose objects are equivalent to homotopy fixed points of G-actions. Indeed, the terminal object $*$ in $\mathfrak{X}_{/BG}$ is id_{BG}, so a section is manifestly an object in $(\mathfrak{X}_{/BG})_*$. To recover the G-action (which is necessarily one that admits a homotopy fixed point), we simply follow the functor $(\mathfrak{X}_{/BG})_* \to \mathfrak{X}_{/BG}$ that forgets the point, while to recover the underlying fixed point $* \to X$, we follow the functor $(b_G^*)_* : (\mathfrak{X}_{/BG})_* \to \mathfrak{X}_*$ induced by $b_G^* : \mathfrak{X}_{/BG} \to \mathfrak{X}$ (*i.e.* pullback along the basepoint $b_G : * \to BG$).

As well as recovering the individual homotopy fixed point corresponding to an object in $(\mathfrak{X}_{/BG})_*$, we can recover the space (not smooth space) of homotopy fixed points of a given action as the fibre of the functor $(\mathfrak{X}_{/BG})_* \to \mathfrak{X}_{/BG}$ above a G-action $X/\!\!/G \to BG$. Indeed, this is equivalent via §2.1.2 of [Lur09a] to the space $\mathfrak{X}(*, X^{hG})$.

It is sometimes useful to exploit a relation, described in Appendix B.1, between the category $(\mathfrak{X}_{/BG})_*$ and the topos $\mathfrak{Fun}(\Delta^1, \mathfrak{X})_{/\mathrm{id}_{BG}}$. Since the latter is evidently equivalent to the topos of actions on objects given by arrows in $\mathfrak{X}$ by the group given by looping the

[5] Note that $\mathfrak{X}_*$ is not a topos (unless it is trivial), so the usual theorems for topoi do not apply.

Definition 5.1.3. *For a topological group G, the category G-$\mathtt{Eta}_X$ is defined to be the category whose objects are triples $(E, p, \mathcal{E})$ where (E, p) is an étalé space over X and $\mathcal{E} = (\{E_g\}_{g\in G}, \{\mathcal{E}_g\}_{g\in G})$ is a partial action of G on E, and whose morphisms between $(E, p, \mathcal{E})$ and $(E', p', \mathcal{E}')$ are étalé morphisms $f : E \to E'$ which satisfy $f(E_g) \subseteq E'_g$ and the analogous commutative diagram to 10.*

We now let G be a Lie group, corresponding to the symmetry group of our system. The corresponding category of étalé spaces is G^d-$\mathtt{Eta}_X$, where the topological group G^d is the group G equipped with the discrete topology. Our functors Γ and J^r can then be modified to account for partial actions as follows.[9]

Definition 5.1.4. *The* equivariant local sections functor $\Gamma : G\text{-}\mathtt{Fib}_X \to G^d\text{-}\mathtt{Eta}_X$ *takes $(Y, \pi, \mathcal{Y})$ to $(\Gamma Y, \Gamma\pi, \Gamma\mathcal{Y})$, with $\Gamma\mathcal{Y} := (\{\Gamma Y_g\}_{g\in G^d}, \{\Gamma\mathcal{Y}_g\}_{g\in G^d})$ and*

$$\Gamma Y_g := \{[\beta]_x \in \Gamma Y \mid \beta(x) \in Y_g \ \forall x \in \mathrm{dom}(\beta), \pi \circ Y_{g^{-1}} \circ \beta \textit{ is an open embedding}\}, \tag{11}$$
$$\Gamma\mathcal{Y}_g : \Gamma Y_{g^{-1}} \to \Gamma Y_g : [\beta]_g \mapsto [Y_g \circ \beta \circ h_{g,\beta}^{-1}]_{h_{g,\beta}(x)}, \tag{12}$$

where $h_{g,\beta}$ is the map defined by $\pi \circ Y_g \circ \beta$, but with its codomain restricted to be its image. The functor Γ takes the morphism $f : Y \to Y'$ to $f : \Gamma Y \to \Gamma Y' : [\alpha]_x \mapsto [f \circ \alpha]_x$.

Definition 5.1.5. *The* equivariant rth-jet functor $J^r : G\text{-}\mathtt{Fib}_X \to G\text{-}\mathtt{Fib}_X$ *takes $(Y, \pi, \mathcal{Y})$ to $(J^r Y, \pi^r, J^r\mathcal{Y})$, with $J^r\mathcal{Y} := (\{J^r Y_g\}_{g\in G}, \{J^r\mathcal{Y}_g\}_{g\in G})$ and*

$$J^r Y_g = \{j^r_x\beta \in J^r Y \mid \beta(x) \in Y_g \ \forall x \in \mathrm{dom}(\beta), \pi \circ Y_{g^{-1}} \circ \beta \textit{ is an open embedding}\}, \tag{13}$$
$$J^r\mathcal{Y}_g : J^r Y_{g^{-1}} \to J^r Y_g : j^r_x\beta \mapsto j^r_{h_{g,\beta}(x)}(Y_g \circ \beta \circ h_{g,\beta}^{-1}), \tag{14}$$

where again, $h_{g,\beta}$ is the map defined by $\pi \circ Y_g \circ \beta$, but with its codomain restricted to its image. The functor J^r takes the morphism $f : Y \to Y'$ to $J^r f : J^r Y \to J^r Y' : j^r_x\alpha \mapsto j^r_x(f \circ \alpha)$.

The functors Γ and J^r preserve the same properties listed in Lemmas 2.3.4 and 2.3.5. In addition, we have the following

Lemma 5.1.6. *Say a morphism between $(Z, \zeta, \mathcal{Z})$ and $(Y, \pi, \mathcal{Y})$ in G-$\mathtt{Fib}_X$ is an* embedding of partial actions *if the underlying fibred morphism $\iota : Z \to Y$ is an embedding such that $Z_g = \iota^{-1}(Y_g)$ for all $g \in G$, along with the analogous statement for G-$\mathtt{Eta}_X$. The functors J^r and Γ preserve embeddings of partial actions. (Proof: Appendix B)*

Returning to natural transformations, we have the following

Proposition 5.1.7. *The maps $\pi^{r,l} : J^r Y \to J^l Y$ form a natural transformation of functors $G\text{-}\mathtt{Fib}_X \to G\text{-}\mathtt{Fib}_X$. The maps $j^r : \Gamma Y \to \Gamma J^r Y$ form a natural transformation in of functors $G\text{-}\mathtt{Fib}_X \to G^d\text{-}\mathtt{Eta}_X$. (Proof: Appendix B)*

We now go on to constraints, which we express by a single

Theorem 5.1.8. *The results given in §3 hold with the categories $\mathtt{Fib}_X$ and $\mathtt{Eta}_X$ replaced with G-$\mathtt{Fib}_X$ and G^d-$\mathtt{Eta}_X$, with the functors replaced by their corresponding equivariant versions defined in 5.1.4 and 5.1.5, and with 'embeddings' replaced with 'embeddings of partial actions'. (Proof: Appendix C)*

[9] The proof that these functors are well defined is given in Appendix B.

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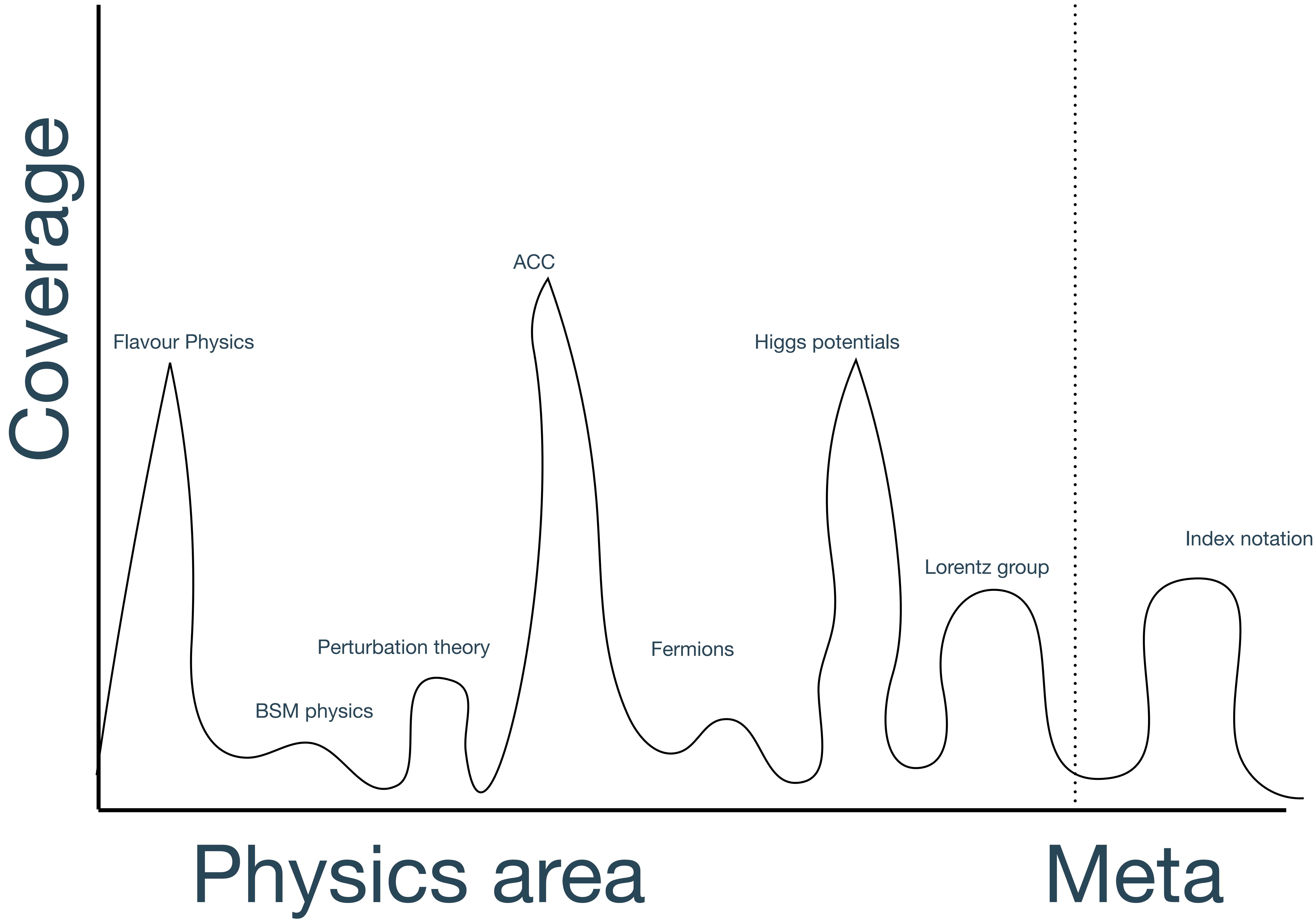
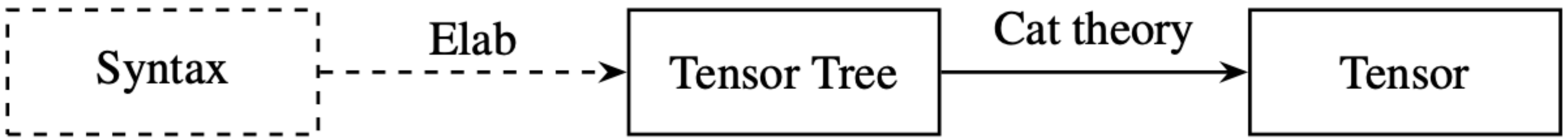


What is the current progress?

Index notation



$$\sigma_{\nu}^{\alpha\beta}\sigma^{\nu\alpha'\beta'} = 2\epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'}$$

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{pauliCo | ν α β ⊗ pauliContr | ν α' β' = 2 •ₜ εL | α α' ⊗ εR | β β'}ᵀ
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How you can get involved!

- Documentation
- Informal lemmas
- Golfing
- Infrastructure
- Promotion.

Thank you!

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